

Colorability Saturation Games

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Abstract

We consider the following two-player game: Maxi and Mini start with the empty graph on n vertices and take turns, always adding one additional edge to the graph such that the chromatic number is at most k , where $k \in \mathbb{N}$ is a given parameter. The game is over when the graph is saturated and no further edge can be inserted. Maxi wants to maximize the length of the game while Mini wants to minimize it. The score $s(n, k)$ denotes the final number of edges in the graph, given that both players played optimally.

This colorability game belongs to the general class of saturation games which aroused interest in the field of combinatorial games during the last years, as they exhibit simple yet challenging problems. The analysis of the described colorability saturation game has been initiated recently by Hefetz et al. [European J. of Comb., Vol. 51(C), 2016]. In this paper, we provide almost-matching lower and upper bounds on the score $s(n, k)$ of the game. In addition, we consider the special case $k = 4$ for which we prove $s(n, 4) = n^2/3 + \mathcal{O}(n)$.

1 Introduction

A typical problem in extremal graph theory is to determine how many edges a graph on n vertices can have without fulfilling a given monotone property \mathcal{P} . A graph $G = (V, E)$ is *saturated* with respect to \mathcal{P} if G does not satisfy \mathcal{P} , but for any $e \in \binom{[n]}{2} \setminus E$ it holds $G \cup \{e\} \in \mathcal{P}$. Then the Turán number $ex(n, \mathcal{P})$ is the maximal number of edges of a \mathcal{P} -saturated graph on n vertices. On the other hand, the saturation number $sat(n, \mathcal{P})$ denotes the minimal possible size of a graph on n vertices which is saturated w.r.t. \mathcal{P} . For a general survey on saturation numbers see [4].

Saturation games are a class of combinatorial games which are closely related to saturated graphs. For a given monotone graph property \mathcal{P} , the saturation game is played as follows: Two players Maxi and Mini start with the empty graph on n vertices. They take turns, always extending the current graph G with some additional edge e such that $G \cup \{e\}$ does not satisfy \mathcal{P} . At some point, every free edge is forbidden, thus the obtained graph G_{end} is saturated w.r.t. \mathcal{P} . Then, the game stops. Mini aims to minimize the number of edges in G_{end} (that is, Mini wants that the game is over as soon as possible), while Maxi's goal is to maximize the number of edges in G_{end} in order to extend the game process. The score of the game, denoted by $s(n, \mathcal{P})$, is the total number of edges in G_{end} when both players apply optimal strategies. In general, the

score can depend on the identity of the first player. However, in this paper we don't specify who starts as all statements will hold for both cases. Clearly, for every monotone property \mathcal{P} we have

$$sat(n, \mathcal{P}) \leq s(n, \mathcal{P}) \leq ex(n, \mathcal{P}),$$

which connects saturation games to the well-studied saturation and Turán numbers of graphs. We see that if the saturation and the Turán numbers of the studied property are the same (for example if \mathcal{P} = "being non-planar"), then the score $s(n, \mathcal{P})$ can be directly determined.

It turned out that analyzing saturation games is both interesting and challenging, as the players not only follow their own strategy and play against the adversary, but additionally the two players have *different* goals. A player wants to create a certain graph structure which ensures a short (resp. long) game, but the more extreme this structure is, the easier the opponent can play against it. A good strategy should (a) be resistant against attacks, and (b) make sure that G_{end} will be sparse (resp. dense). Finding optimal or almost-optimal strategies can be surprisingly hard and often requires tedious case distinctions. Thus, the asymptotic expression of the score is only known for a few particular games.

In the following we give a brief overview about some existing results. Let \mathcal{C}_k be the property of being k -connected and spanning, and let \mathcal{PM} be the property of admitting a perfect matching. Carraher, Kinnersley, Reiniger, and West proved $s(n, \mathcal{C}_1) = \binom{n-2}{2} + 1$ for the connectivity game [2]. Shortly afterwards, Hefetz, Krivelevich, Naor, and Stojaković [6] generalized this result and asserted $s(n, \mathcal{C}_k) \geq \binom{n}{2} - 5kn^{3/2}$. In the same paper, they proved $s(n, \mathcal{PM}) \geq \binom{n-4}{2}$ and further results on matching games. Additional saturation games have been studied by Lee and Riet [8], as well as variants on directed graphs [7].

The most famous example of saturation games is the triangle-free game. In this game, the considered monotone property is \mathcal{K}_3 = "containing a triangle". It is well-known that $ex(n, \mathcal{K}_3) = \lfloor n^2/4 \rfloor$ and $sat(n, \mathcal{K}_3) = n - 1$ (see e.g. [3]). In [5] and [9], Füredi, Reimer, and Seress proved a lower bound of $(\frac{1}{2} + o(1))n \log_2 n$ on the score of this game, and cite Erdős who has given an upper bound of $n^2/5$ in personal communication. However, the proof of this statement is lost. The currently best-known upper bound is $\frac{26}{121}n^2 + o(n^2)$ by Biró, Horn and Wildstrom [1], a small improvement of the trivial upper-bound. Closing the large gap between $\Omega(n \log n)$ and $\mathcal{O}(n^2)$ is still a big open problem, and the current understanding of this game is rather dissatisfying.

In this paper we focus on the property

$$\chi_{>k} = \text{"having chromatic number at least } k + 1\text{"}.$$

In other words, Mini and Maxi are only allowed to place edges such that the graph remains k -colorable. We will use the abbreviation $s(n, k) := s(n, \chi_{>k})$ for the score of these games. Let G_{end} be a graph on $n > k$ vertices which is saturated w.r.t. $\chi_{>k}$. Obviously, the graph G_{end} must be a complete k -partition, thus

$$sat(n, \chi_{>k}) = (k-1)(n-1) - \binom{k-1}{2}$$

and

$$ex(n, \chi_{>k}) = (1 - 1/k + o(1)) \binom{n}{2}.$$

This gives us first bounds on the score of the corresponding saturation game.

Let us start with a short description of the case $k = 2$. Here, the two players are forced to keep the graph bipartite. If n is even, Maxi can play such that always after her moves, every component of the current graph is balanced, and therefore G_{end} will be perfectly balanced. This argument leads to

$$s(n, 2) = ex(n, \chi_{>2}) = \left\lfloor \frac{n}{4} \right\rfloor,$$

a formal proof is given in [2]. Thus, Mini has no power in this saturation game. However, things get more interesting and involved if $k > 2$. The authors of [6] showed $s(n, 3) \leq 21n^2/64 + \mathcal{O}(n)$, and we see that now, Mini has some influence on the game process. Furthermore, in the same paper they proved the following general statement for colorability saturation games.

Theorem 1.1 (Theorem 1.5 in [6]). *Let $k \in \mathbb{N}$ and let n be sufficiently large. Then there exists a constant $C > 0$ such that*

$$s(n, k) \geq \binom{n}{2} \left(1 - \frac{C \log k}{k} \right).$$

The proof of this result requires a relatively large constant C which makes the statement impractical for small integers k .

As our main contribution, we provide almost-matching lower and upper bounds on $s(n, k)$ which hold also for small parameters k and demonstrate how $s(n, k)$ depends on the parameter k . This enhances the understanding of colorability saturation games in particular and saturation games in general. Our first result is a general lower bound on the score and improves Theorem 1.1.

Theorem 1.2. *Let $k \geq 3$ and $n > k$. Then*

$$s(n, k) \geq \binom{n}{2} \left(1 - \frac{1}{\lceil k/2 \rceil} \right) \geq \binom{n}{2} \left(1 - \frac{2}{k} \right).$$

Note that for $k = k(n)$ and $n \rightarrow \infty$, the theorem and the Turán number $ex(n, \chi_{>k})$ together imply

$$s(n, k) = \binom{n}{2} - \Theta\left(\frac{n^2}{k}\right).$$

Next, we provide a general upper bound which proves in particular that if Mini follows an optimal strategy, the number of missing edges at the end of the game is by a constant factor larger than in a balanced complete k -partite graph, thus the upper bound given by the Turán number is not tight.

Theorem 1.3. *Let $k \geq 4$ and $n > k$. Then*

$$s(n, k) \leq \binom{n}{2} \left(1 - \frac{1}{k - \lfloor (k-1)/3 \rfloor} \right) + kn \leq \binom{n}{2} \left(1 - \frac{3}{2k+3} \right) + kn.$$

In particular, if k is fixed and $n \rightarrow \infty$, then

$$s(n, k) \leq \binom{n}{2} \left(1 - \frac{3}{2k+3} + o(1) \right).$$

The provided lower and upper bounds on $s(n, k)$ are matching up to a constant factor in the term counting the missing edges. It remains an interesting problem to determine the correct constant. With our last result we analyze the special case $k = 4$, where it turns out that the upper-bound given by Theorem 1.3 is tight.

Theorem 1.4. *Let $n \geq 5$. Then $s(n, 4) = n^2/3 + \mathcal{O}(n)$.*

We prove Theorem 1.2 and Theorem 1.3 by using potential functions which are closely related to the density of induced subgraphs of G . They allow us to define strategies in a general, abstract way such that we only need to deal with a reasonable number of case distinctions. To the best of our knowledge, this is a new approach for studies of saturation games.

Below in Section 2 we introduce the notations, collect auxiliary lemmas and describe general aspects of our strategies on a high-level. Then in Section 3 we provide a general strategy for Maxi, whose analysis will give Theorem 1.2. In Section 4 we describe a strategy for Mini in order to obtain Theorem 1.3. Afterwards in Section 5 we propose a more detailed strategy for Maxi for the special case $k = 4$ which yields Theorem 1.4. Finally the last section contains some concluding remarks and open problems.

2 Preliminaries

For a given parameter $k \in \mathbb{N}$ we consider the colorability saturation game on a set of n vertices. We always think of the game as a process in time, thus $G(t)$ denotes the resulting graph after the first t rounds. That is, $G(0)$ is the empty graph by definition, afterwards $G(t)$ contains exactly t edges, until at time t_{end} the graph $G_{end} = G(t_{end})$ is a complete k -partition. It turns out that whenever we study the game from the perspective of one specific player, the process has two phases: In the first phase, the player tries to build up a certain graph structure which covers all vertices. In the second phase, the player can insert edges arbitrarily until the graph is saturated.

The strategies will describe how the desired graph structure should be created during the first phase. For this purpose, we will require that during the game process, our player maintains two lists W and C of vertices: The list $C = C(t)$ of *completed* vertices contains, informally speaking, the vertices where the desired structure is already present, and vice-versa the list $W = W(t)$ contains the vertices in the *working* part where the player still wants to play certain edges. This yields an artificial partition

$$V = W(t) \dot{\cup} C(t)$$

on which the strategies will be based. We will start with $W(0) := V$ and $C(0) := \emptyset$. Afterwards, the strategies will precisely define how the lists W and C should be updated during the game.

Let A and B be two disjoint subsets of vertices in a graph G , and denote by $E[A]$ the set of edges in the subgraph induced by A and by $E[A, B]$ the set of edges which have one incident vertex in both A and B . Then we define

$$\phi(A, B) := |E[A]| + |E[A, B]|.$$

Typically, we will consider a point in time t , $A \subseteq W(t)$ and $B = C(t)$. In this case we write

$$\phi(A, t) := \phi(A, C(t)).$$

This potential function ϕ measures the density in different parts of the current graph and will be crucial for our strategies later on.

After having introduced the most important notations, we start collecting some general lemmas which will be crucial for our strategies. Recall that G_{end} is always a complete k -bipartite graph. The following technical lemma analyzes the total number of edges in a complete k -partite graph which is created by merging two smaller k -partite graphs together.

Lemma 2.1. *Let $k \geq 2$ and let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two complete k -partite graphs with partitions $V_1 = \dot{\cup}_{i=1}^k V_{1,i}$, resp. $V_2 = \dot{\cup}_{i=1}^k V_{2,i}$. Assume w.l.o.g. that for all $i < k$ it holds $|V_{1,i}| \geq |V_{1,i+1}|$ and $|V_{2,i}| \geq |V_{2,i+1}|$. For a permutation $\sigma \in \mathcal{S}_k$, let $G_\sigma = (V, E_\sigma)$ be the complete k -bipartite graph with vertex set $V = V_1 \cup V_2$ and partitions $V = \dot{\cup}_{i=1}^k (V_{1,i} \cup V_{2,\sigma(i)})$.*

Then, the total number of edges $|E_\sigma|$ in G_σ is minimized with $\sigma \equiv id$, and maximized if $\sigma(i) = k - i + 1$ for all $1 \leq i \leq k$.

Proof. Clearly, H_1 and H_2 are contained in G as induced subgraphs, and we have $|E[G[V_1]]| = |E_1|$ and $|E[G[V_2]]| = |E_2|$ independently of the permutation σ . Hence we need to prove that the given permutations minimize (resp. maximize) the number of edges in $E[V_1, V_2]$. For this we observe that

$$|E[V_1, V_2]| = \sum_{i=1}^k |V_{1,i}| \cdot (|V_2| - |V_{2,\sigma(i)}|) = |V_1| \cdot |V_2| - \sum_{i=1}^k |V_{1,i}| \cdot |V_{2,\sigma(i)}|.$$

Suppose that σ is not the identity, and let i be the smallest index for which $\sigma(i) \neq i$. We create a new permutation σ' as follows: We put $\sigma'(i) := i$, $\sigma'(\sigma^{-1}(i)) := \sigma(i)$, and for all other j we keep $\sigma'(j) := \sigma(j)$. We claim that with σ' instead of σ , the number of edges between V_1 and V_2 decreases by $\Delta \geq 0$. Indeed,

$$\begin{aligned} \Delta &= |V_{1,i}| \cdot |V_{2,i}| + |V_{1,\sigma^{-1}(i)}| \cdot |V_{2,\sigma(i)}| - |V_{1,i}| \cdot |V_{2,\sigma(i)}| - |V_{1,\sigma^{-1}(i)}| \cdot |V_{2,i}| \\ &= (|V_{1,i}| - |V_{1,\sigma^{-1}(i)}|) \cdot (|V_{2,i}| - |V_{2,\sigma(i)}|) \\ &\geq 0. \end{aligned}$$

By repeating the argument, it follows that $\sigma \equiv id$ minimizes the number of edges between V_1 and V_2 and also the total number of edges in G_σ .

The second statement for maximizing $|E_\sigma|$ can be proven similarly. \square

The next lemma is standard and says that if C is k -colorable and every subset $A \subseteq W$ satisfies a density-condition (which is closely related to the potential ϕ), then any proper k -coloring of C can be extended to W . Thereby, the set C is also allowed to be empty.

Lemma 2.2. *Let $k \in \mathbb{N}$ and $G = (V, E)$ be a graph such that $V = W \dot{\cup} C$, the induced subgraph $G[C]$ is k -colorable and for every non-empty subset $A \subseteq W$ it holds $2|E[A]| + |E[A, C]| < k \cdot |A|$. Then G is k -colorable.*

Proof. Let $m = |W|$. For every ordering v_1, v_2, \dots, v_m of the vertices in W we can define $W_0 := W$ and $W_i := W_{i-1} \setminus \{v_i\}$ for $i \in \{1, \dots, m\}$. We claim that there exists an ordering W such that for all $i \in \{1, \dots, m\}$, the vertex v_i has at most $k - 1$ neighbors in $C \cup W_i$. Indeed, by assumption we have

$$\sum_{v \in W} \deg(v) = 2|E[W]| + |E[W, C]| < k|W|,$$

thus there exists a vertex $v_1 \in W$ with degree at most $k - 1$. Now let $1 \leq i < m$ and assume by induction that we have found vertices v_1, \dots, v_i with the desired property. Because the set W_i satisfies

$$2|E[W_i]| + |E[W_i, C]| < k \cdot |W_i|,$$

there exists a vertex $v_{i+1} \in W_i$ with at most $k - 1$ neighbors in $W_i \cup C$. With this vertex we can extend the ordering to v_1, \dots, v_{i+1} as desired.

Given this ordering of V , it is easy to find a proper k -coloring of G : By assumption there exists a proper coloring of $G[C \cup W_m] = G[C]$. Then we can inductively extend a proper k -coloring of $G[C \cup W_i]$ to $G[C \cup W_{i-1}]$ as, by construction, v_i has at most $k - 1$ neighbors in $C \cup W_i$ and there exists a free color for v_i . At last, this gives a proper coloring of $G[C \cup W_0] = G$. \square

We finish this preliminary section with a technical lemma which compares the potential ϕ to a given function f on the vertices. Informally speaking, think of $f(v)$ as measuring the progress of the considered player on a vertex $v \in V$. Then the lemma describes how ϕ changes compared to f if one additional edge is added to the graph.

Lemma 2.3. *Let $G = (V, E)$, $V = W \dot{\cup} C$ and let $f : W \rightarrow \mathbb{N} \cup \{0\}$ be a function such that for every set $A \subseteq W$ it holds $\sum_{v \in A} f(v) \geq \phi(A, C)$. Let $e \notin E$, $G' := G \cup \{e\}$, and for G' consider*

$$\mathcal{D} := \{A \subseteq W : \sum_{v \in A} f(v) < \phi(A, C)\}.$$

If \mathcal{D} is non-empty, then G' has the following two properties:

- (a) $A_0 := \cap_{A \in \mathcal{D}} A$ is non-empty and itself contained in \mathcal{D} .
- (b) For all $A \subseteq W \setminus A_0$ we have

$$\sum_{v \in A} f(v) \geq |E[A]| + |E[A, C \cup A_0]|.$$

Proof. First we observe that the empty set can not be in \mathcal{D} as $\phi(\emptyset, C) = 0$. Next assume that \mathcal{D} is non-empty and let $A_1, A_2 \in \mathcal{D}$. We claim that $A_1 \cap A_2 \in \mathcal{D}$. By assumption, in G we had $\sum_{v \in A_1} f(v) \geq \phi(A_1, C)$. The potential $\phi(A_1, C)$ can increase by at most one when the additional edge $e = \{x, y\}$ is added, thus in G we had

$$\sum_{v \in A_1} f(v) = \phi(A_1, C).$$

Notice that the same holds for A_2 as well. For the graph G it follows

$$\begin{aligned}\phi(A_1, C) + \phi(A_2, C) &= \sum_{v \in A_1} f(v) + \sum_{v \in A_2} f(v) = \sum_{v \in A_1 \cup A_2} f(v) + \sum_{v \in A_1 \cap A_2} f(v) \\ &\geq \phi(A_1 \cup A_2, C) + \phi(A_1 \cap A_2, C) \\ &\geq \phi(A_1, C) + \phi(A_2, C),\end{aligned}$$

where the first inequality follows by our assumption on G and second inequality follows from the fact that every edge is counted in $\phi(A_1 \cup A_2, C) + \phi(A_1 \cap A_2, C)$ at least as often as in $\phi(A_1, C) + \phi(A_2, C)$, which can be observed by a simple case analysis. However, the above inequality chain implies that we have equality everywhere, and in particular in G it holds

$$\sum_{v \in A_1 \cap A_2} f(v) = \phi(A_1 \cap A_2, C).$$

We know that both $\phi(A_1, C)$ and $\phi(A_2, C)$ increased by one when adding the additional edge $e = \{x, y\}$. Then either both $x, y \in A_1 \cap A_2$ or one endpoint is in $A_1 \cap A_2$ and the other in C . In both cases, the edge e contributes to $\phi(A_1 \cap A_2, C)$. As we had $\sum_{v \in A_1 \cap A_2} f(v) = \phi(A_1 \cap A_2, C)$ before adding the new edge e , we deduce that $A_1 \cap A_2 \in \mathcal{D}$ in G' . We can repeat the argument for any two sets $A_1, A_2 \in \mathcal{D}$ and conclude that $A_0 = \cap_{A \in \mathcal{D}} A$ is itself in the family \mathcal{D} . This proves (a).

For (b), we observe that $\phi(A_0, C)$ could increase by at most one when adding e , thus in G we had $\phi(A_0, C) = \sum_{v \in A_0} f(v)$. Let $A \subseteq W \setminus A_0$. We observe that in G , the edges counted in $|E[A]| + |E[A, C \cup A_0]|$ are precisely the edges which contribute to $\phi(A \cup A_0, C) - \phi(A_0, C)$. Recall that by assumption we had $\sum_{v \in A \cup A_0} f(v) \geq \phi(A \cup A_0, C)$. Putting things together, we then arrive at

$$\begin{aligned}|E[A]| + |E[A, C \cup A_0]| &= \phi(A \cup A_0, C) - \phi(A_0, C) \\ &\leq \sum_{v \in A \cup A_0} f(v) - \sum_{v \in A_0} f(v) = \sum_{v \in A} f(v).\end{aligned}$$

This inequality holds for both G and G' because $e \notin E[A] \cup E[A, C \cup A_0]$. This finishes the proof of the lemma. \square

3 Lower Bound Strategy

In this section we describe a general strategy for Maxi which implies Theorem 1.2. Maxi wants to play such that no color class of the saturated graph G_{end} is too large. This ensures that G_{end} has many edges and thus elongates the game. For this purpose, she uses her moves to create a collection of vertex-disjoint cliques. Because every independent set can have at most one vertex per clique, this forbids large color classes in G_{end} . The general lower bound follows directly from the following lemma.

Lemma 3.1. *Let $m \geq 2$ be an integer and $k \geq 2m - 1$. Then in the colorability saturation game with k colors, Maxi has a strategy such that in G_{end} , the vertex set is covered by $\lceil \frac{n}{m} \rceil - 1$ vertex-disjoint cliques, all having size m , and one additional clique including the remaining vertices.*

Proof of Theorem 1.2. Let $n > k$ and $m = \lceil \frac{k}{2} \rceil$. G_{end} is a complete k -partite and contains all $\binom{n}{2}$ edges except those inside one of the k partitions. By Lemma 3.1, Maxi has a strategy such that G_{end} contains $\lceil \frac{n}{m} \rceil$ vertex-disjoint cliques as induced subgraphs which cover V completely, where every clique except the last one contains m vertices. All vertices of such a clique must belong to different partitions of the saturated graph G_{end} .

Now we do a worst-case analysis in order to lower-bound the total number of edges in G_{end} . By Lemma 2.1, this number is the smallest if G_{end} contains m classes of size $\frac{n}{m}$, all sharing one vertex with every clique given by Lemma 3.1. Then the total number of missing edges in G_{end} is at most

$$m \cdot \binom{\frac{n}{m}}{2} = \frac{n}{2} \left(\frac{n}{m} - 1 \right) \leq \binom{n}{2} \frac{1}{m},$$

which proves the statement. \square

Proof of Lemma 3.1. As discussed in the previous section, during the game Maxi maintains a list C of vertices in the "completed part" and a list W of vertices in the "working part". The lists $C = C(t)$ and $W = W(t)$ change during the time. For abuse of notation, we neglect the time-dependence of W and C . Maxi starts with $\emptyset \rightarrow C$ and $V \rightarrow W$. Whenever Maxi is not explicitly required to update the sets W and C , we implicitly assume that the lists remain the same until further notice.

Furthermore, we need a second classification of the vertex set. Before the game starts, Maxi splits the vertex set V into groups of always m vertices. Formally, we get a second partition $V = \dot{\cup}_i V_i$ with $|V_i| = m$ for all i . The last group may contain less vertices if $\frac{n}{m}$ is not an integer. For $v \in V_i$ we define $\alpha(v)$ as the number of neighbours inside its own set V_i , and for every subset $A \subseteq V$ we put $\alpha(A) := \sum_{v \in A} \alpha(v)$. Initially, $\alpha(v) = 0$ for all $v \in V$. Afterwards, the values $\alpha(v)$ increase during the game process. We call a vertex $v \in V_i$ *full* if $\alpha(v) = |V_i| - 1$. Moreover, for all points in time t we define

$$\mathcal{D}(t) := \{A \subseteq W : \alpha(A) < \phi(A, t)\},$$

where we use the potential ϕ as defined in Section 2. Clearly, it holds $\mathcal{D}(0) = \emptyset$.

Having introduced all notations, we start presenting our strategy for a move of Maxi at time t . The strategy consists of two parts: First, if necessary, Maxi updates the sets W and C , as a first answer to the precedent move of Mini.

(U1) If $\mathcal{D}(t-1)$ is non-empty and all $v \in A_0 := \arg \min_{A \in \mathcal{D}(t-1)} |A|$ are full, update $C \cup A_0 \rightarrow C$ and $W \setminus A_0 \rightarrow W$.

If (U1) does not apply, then the sets W and C don't change. Otherwise, updating C and W changes also the set $\mathcal{D}(t-1)$, because this set is defined via C and the graph still contains $t-1$ edges. Later on we will observe that applying (U1) makes $\mathcal{D}(t-1)$ empty. The potential execution of (U1) is followed by the real move of Maxi at time t , where she tries to place a new edge. For the real move, we give the following four rules.

(S1) If W is empty or all vertices in W are full, and $G(t-1)$ is saturated, the game stops.

- (S2) If W is empty or all vertices in W are full, but $G(t-1)$ is not saturated, play arbitrarily (but respecting the colorability constraint), and put $\emptyset \rightarrow W$ and $V \rightarrow C$.
- (S3) If $\mathcal{D}(t-1)$ is empty and not every vertex in W is full, insert a free edge inside any set V_i .
- (S4) If there exists a non-full vertex $v \in A_0 := \arg \min_{A \in \mathcal{D}(t-1)} |A|$ and $\mathcal{D}(t-1)$ is non-empty, connect v with another vertex in its group V_i .

Note that in (S3) or (S4), Maxi does not change C and W . We observe that the strategy ensures that C never contains non-full vertices, as Maxi can extend the set of completed vertices only with (U1) where all transferred vertices are full. We need to prove that the strategy is well-defined and that it leads to the desired structure. We do this with the following claim, whose proof is deferred to the end of this section.

Claim 3.2. *The proposed strategy is well-defined and can be applied by Maxi without violating the colorability constraint of the game (n, χ_k) . Moreover, if $G(t-1)$ is not saturated and Maxi plays at time t , the following invariants hold:*

- (I1) $\mathcal{D}(t)$ is empty.
- (I2) If there exists a non-full vertex $v \in W$ in the graph $G(t)$, then $G(t)$ is not saturated.

With this claim we can finish the proof of the lemma as follows: Suppose Maxi follows the proposed strategy. Whenever Mini plays at time $t-1$, by Claim 3.2 Maxi has an appropriate answer. Assume that after doing the potential update (U1), not every vertex in W is full. Then (S1) and (S2) are forbidden and Maxi applies one rule of (S3) and (S4). In both cases, Maxi can add an edge to the graph such that $G(t)$ is k -colorable, thus $G(t-1)$ was not saturated. Hence, after Mini's move at time $t-1$, the game can stop only if all $v \in W$ are full or if W is empty. On the other hand, if Maxi follows the strategy, by (I2) she can only make $G(t)$ saturated if every vertex in W is full. We conclude that at time t_{end} , either $W = \emptyset$ or every $v \in W$ is full. However, we already know that all $v \in C$ are full as well. Thus, at t_{end} every vertex $v \in V$ is full, and by definition this implies that every set V_i is a clique of size $|V_i|$. \square

Proof of Claim 3.2. We prove the statement by induction over all moves of Maxi. For the empty graph $G(0)$, clearly $\mathcal{D}(0) = \emptyset$. Assume now by induction that either it is the first move of Maxi (which will give the induction basis) or that she could apply the strategy until time t . We shall first prove that always one of the four rules applies. We start by considering the potential updating (U1). Suppose $\mathcal{D}(t-1)$ is non-empty before Maxi applies (U1). Then $t > 1$, and by induction it holds $\mathcal{D}(t-2) = \emptyset$. We apply Lemma 2.3 (a) with $f = \alpha$ and $G = G(t-2)$ to see that $A_0 := \arg \min_{A \in \mathcal{D}(t-1)} |A|$ is uniquely defined and exactly the intersection of all elements of $\mathcal{D}(t-1)$. If all vertices in A_0 are full, we apply Lemma 2.3 (b) with $G' = G(t-1)$. It follows that after updating C and W with (U1), the set $\mathcal{D}(t-1)$ is already empty, and always one rule of (S1)-(S4) applies. Clearly, for (S2)-(S4) there always exists a free edge matching to the rule. For well-definiteness it remains to show that such an edge keeps the graph k -colorable.

Suppose $G(t-1)$ is not saturated. Let $e = \{x, y\}$ be the edge which Maxi wants to add after the potential updating (U1). We let her play this edge, put $G(t) = G(t-1) \cup \{e\}$ and assume for now that $\mathcal{D}(t)$ is empty. For the rule (S2), it is clear that $G(t)$ is k -colorable. Otherwise, we first observe that since every vertex in C is full, $e \in W$. Clearly, $G[C]$ is k -colorable. Assume for now that (I1) holds for $G(t)$, that is, $\mathcal{D}(t) = \emptyset$. For every non-empty set $A \subseteq W$ it follows

$$2|E[A]| + |E[A, C]| \leq 2\phi(A, t) \leq 2\alpha(A) \leq 2(m-1)|A| < (2m-1)|A|.$$

Applying Lemma 2.2 and using $k \geq 2m-1$, we see that indeed, $G(t)$ is k -colorable and Maxi can play the desired edge.

Next, we prove Invariant (I1). Consider the situation after running (U1). If Maxi uses (S2), the set W is empty and there is nothing to show. Whenever Maxi plays (S3), we have already $\mathcal{D}(t-1) = \emptyset$. Both $x, y \in W$, and both $\alpha(x)$ and $\alpha(y)$ increase by one with the edge e as x and y are part of the same set V_i . Hence for every set $A \in W$, we have $\phi(A, t) - \phi(A, t-1) = 1$ if and only if both $x, y \in A$. But in this case, $\alpha(A)$ increases by two while e is added. Then $A \notin \mathcal{D}(t)$, and we see that $\mathcal{D}(t) = \emptyset$ after applying (S3).

At last suppose Maxi applies (S4). Here we have $t \geq 2$, and by Lemma 2.3 (a) it holds $A_0 := \cap_{A \in \mathcal{D}(t-1)}$. Let $A \in \mathcal{D}(t-1)$. The induction assumption yields $\alpha(A) \geq \phi(A, t-2)$ for the graph $G(t-2)$. Now by definition of (S4), at least one vertex of e is contained in $A_0 \subseteq A$, say x . We distinguish two cases: If both $x, y \in A$, then $\phi(A, t) - \phi(A, t-2) = 2$ as the potential counts both e and the edge which Mini played before. But at the same time, $\alpha(A)$ increased by two as well since both x and y got a new neighbor in their set V_i . On the other hand, if only $x \in A$, then $e \notin E[A]$. Because C contains only full vertices, we have $y \notin C$ and therefore e does not contribute to the potential of the set A . In this case $\phi(A, t) - \phi(A, t-2) = 1$, due to Mini's edge. However, in meantime $\alpha(A)$ increases also by one as $\alpha(x)$ gets one larger. In both cases it follows that the set A is fine again and not contained in $\mathcal{D}(t)$. This holds for every subset $A \subseteq W$, and indeed $\mathcal{D}(t)$ is empty.

The second invariant (I2) is easy: Suppose there exists a vertex $v \in W$ which is not full. Then there exists a free edge $e' = \{x', y'\}$ inside a group V_i . Let $G(t+1) = G(t) \cup \{e'\}$. As (I1) is proven, we know that $\mathcal{D}(t)$ is empty. In this case, the same argument as above for (S3) yields that $\mathcal{D}(t+1) = \emptyset$, and from Lemma 2.2 we deduce that $G(t+1)$ is k -colorable. Thus $G(t)$ was not saturated. \square

4 Upper Bound Strategy

Here we provide a general strategy for the purpose of shortening the game. The main idea behind this strategy for Mini is to create many stars. Roughly speaking, a vertex of degree $n-1$ removes one color from the game, and if Mini can draw many stars, less colors are available for the remaining part of the graph which makes G_{end} sparser. We can deduce Theorem 1.3 directly from the following lemma.

Lemma 4.1. *Let $\ell \in \mathbb{N}$ and $k \geq 3\ell+1$. Then in the colorability saturation game with k colors, Mini has a strategy such that G_{end} contains at least ℓ vertices of degree $n-1$.*

Proof of Theorem 1.3. Let $n > k$, $k \geq 4$ and $\ell := \lfloor \frac{k-1}{3} \rfloor$. The graph G_{end} will be completely k -partite with partition $V = \dot{\cup}_{i=1}^k C_i$. By Lemma 4.1, Mini has a strategy such that at least ℓ vertices have degree $n-1$ in G_{end} . Assume w.l.o.g. that $|C_i| = 1$ for all $i \leq \ell$. Then the total number of missing edges in G_{end} is $\sum_{i=1}^k \binom{|C_i|}{2} = \sum_{i=\ell+1}^k \binom{|C_i|}{2}$. By Lemma 2.1, this number is minimal if all sets $C_{\ell+1}, \dots, C_k$ are equally large. In this worst case, we have $|C_i| = \frac{n-\ell}{k-\ell}$ for all $i \in \{\ell+1, \dots, k\}$. Then we obtain

$$\begin{aligned} s(n, k) &\leq \binom{n}{2} - (k-\ell) \binom{\frac{n-\ell}{k-\ell}}{2} = \binom{n}{2} - \frac{(n-\ell)(n-k)}{2(k-\ell)} \\ &\leq \binom{n}{2} \left(1 - \frac{1}{k-\ell}\right) + \frac{(n-1)(k+\ell)}{2(k-\ell)}. \end{aligned}$$

Furthermore, $\frac{k}{3} \geq \ell \geq \frac{k-3}{3}$ by our choice of ℓ , therefore $\frac{1}{k-\ell} \geq \frac{3}{2k+3}$ and $\frac{k+\ell}{k-\ell} \leq 2$. This proves

$$s(n, k) \leq \binom{n}{2} \left(1 - \frac{3}{2k+3}\right) + nk.$$

□

Proof of Lemma 4.1. Let $\ell \in \mathbb{N}$ and $k \geq 3\ell + 1$. Mini wants to play such that in G_{end} , at least ℓ vertices have degree $n-1$. Our strategy will be such that Mini maintains a list S containing the vertices whose degree should increase to $n-1$ until t_{end} . The strategy will use the artificial partition

$$V = W \dot{\cup} C \dot{\cup} S,$$

where W is again the "working part" and C contains the "completed" vertices. Again, the lists W , C , and S will be frequently updated by Mini.

For every vertex $v \notin S$ we define the *level* $\ell(v) := |N(v) \cap S|$, and for a set $A \subseteq W$ we define $\alpha(A) := \sum_{v \in A} \ell(v)$ and the potential

$$\phi(A, t) = |E[A]| + |E[A, C]|,$$

where $C = C(t)$ is the list of Mini at time t . Note that we don't consider sets A that share vertices with S . Finally, for all points in time t we put

$$\mathcal{D}(t) := \{A \subseteq W : \alpha(A) < \phi(A, t)\}.$$

We start presenting the strategy. Mini starts by initializing $\emptyset \rightarrow S$, $\emptyset \rightarrow C$ and $V \rightarrow W$. Then a move of Mini at time t is again given by an updating-phase and an edge-drawing-phase, similar to our Maxi-strategy in Section 3. Before claiming her own edge, we require that Mini applies repeatedly one of the following update rules for the sets W , C , and S , until no rule is matching any more.

- (U1) If $\mathcal{D}(t-1)$ is non-empty and all $v \in A_0 := \arg \min_{A \in \mathcal{D}(t-1)} |A|$ have level ℓ , update $C \cup A_0 \rightarrow C$ and $W \setminus A_0 \rightarrow W$.
- (U2) If $\mathcal{D}(t-1)$ is non-empty and all $v \in A_0 := \arg \min_{A \in \mathcal{D}(t-1)} |A|$ have level $|S| < \ell$, take an arbitrary vertex $v \in A_0$, update $S \cup \{v\} \rightarrow S$ and $W \setminus \{v\} \rightarrow W$.

- (U3) If $\mathcal{D}(t-1)$ is empty and all $v \in W$ have level $|S| < \ell$, let $v \in A_0$ arbitrarily, update $S \cup \{v\} \rightarrow S$ and $W \setminus \{v\} \rightarrow W$.

The execution of these update rules changes the set $\mathcal{D}(t-1)$ as well because it is defined via W , C , and S . Hence, if Mini executes a sequence of multiple updates, $\mathcal{D}(t-1)$ changes repeatedly until none of the rules applies anymore. After this updating phase, Mini is told to draw her edge at time t according to the following rules.

- (S1) If W is empty or all vertices in W have level ℓ , and $G(t-1)$ is saturated, the game stops.
- (S2) If W is empty or all vertices in W have level ℓ , but $G(t-1)$ is not saturated, play arbitrarily (but respecting the colorability constraint), put $\emptyset \rightarrow W$ and $V \setminus S \rightarrow C$.
- (S3) If $\mathcal{D}(t-1)$ is empty and not every vertex in W has level $|S|$, play an arbitrary edge between W and S .
- (S4) If there exists a vertex v in $A_0 := \arg \min_{A \in \mathcal{D}(t-1)} |A|$ which has not yet level $|S|$ and $\mathcal{D}(t-1)$ is non-empty, connect v with a vertex in S .

Note that in (S3) and (S4), Maxi does not change the sets W , C , and S . We see that as long as Mini can follow this strategy, several properties follow immediately for all points in time t . First of all, C contains only vertices of level ℓ , as Mini can extend this list only when applying (U1), where all transferred vertices already have level ℓ . Next we observe that the set S contains at most ℓ vertices. Moreover, whenever a vertex v is added to S , it is already connected to all previous vertices of S (see (U2) and (U3)), thus $G[S]$ is always a clique of size $|S|$. We shall verify that our proposed strategy is well-defined and valid for Mini. We do this with the following claim which we prove below.

Claim 4.2. *The proposed strategy is well-defined and can be applied by Mini without violating the colorability constraint. Moreover, if $G(t-1)$ is not saturated and Mini plays at time t , the following invariants hold:*

- (I1) $\mathcal{D}(t)$ is empty.
- (I2) If in $G(t)$, there exists a vertex $v \in W$ with level less than ℓ , then $G(t)$ is not saturated.

With Claim 4.2 the proof can be finished as follows: At time t , Mini has an appropriate answer to Maxi's previous move. If there exists a vertex $v \in W$ with level less than ℓ after the updating, Mini draws an edge e by using either (S3) or (S4), and $G(t)$ is still k -colorable. Therefore, after Maxi's move at time $t-1$, the game can stop only if W is empty or contains only vertices of level ℓ . On the other hand, by (I2) of Claim 4.2 Mini as well can finish the game only if after her move, either W is empty or every vertex therein has level ℓ . We have already seen that the vertices in S always form a clique and all vertices in C have level ℓ . Therefore $\ell(v) = \ell$ for all $v \notin S$ and we conclude that at time t_{end} , the set S contains ℓ vertices and all of them have degree $n-1$. \square

Proof of Claim 4.2. We prove the statement by induction over all moves of Mini. First observe that for the start, the empty graph $G(0)$ satisfies both (I1) and (I2). Then we assume that either it is the first move of Mini (which yields the induction basis) or that she could apply the strategy until t . We start by showing that in any case, one rule of (S1)-(S4) applies, and that Mini always finds a free edge $f = \{x, y\}$ matching to the rule.

Assume that $\mathcal{D}(t-1)$ is non-empty. Then $t > 1$ and by induction, $\mathcal{D}(t-2)$ was empty before. We apply Lemma 2.3 (a) with $G = G(t-1)$, $G' = G(t)$ and $f(v) = \ell(v)$ to see that $A_0 = \arg \min_{A \in \mathcal{D}(t-1)} |A|$ is uniquely defined and precisely the intersection of all sets in $\mathcal{D}(t-1)$. If there exists a free edge between A_0 and S , no update rule applies and Mini can directly use (S4). Next suppose that before doing updates, all vertices in A_0 are connected to every vertex in S . Then Mini uses either (U1) or (U2). In the first case, she moves A_0 from W to C . We apply Lemma 2.3 (b) and see that $\mathcal{D}(t-1)$ is empty after the update. In the second case, a vertex $v \in A_0$ is moved from W to S . Note that *before* the update, $v \in A$ for all sets $A \in \mathcal{D}(t-1)$. Hence *after* the update, none of these sets A is a subset of W , and we deduce again that $\mathcal{D}(t-1)$ is empty after the update.

If $\mathcal{D}(t-1)$ is empty and every vertex in S is connected to all $w \in W$, Mini uses update rule (U3). First note that after moving a vertex from W to S , $\mathcal{D}(t-1)$ remains empty. Second notice that this rule might be repeatedly applied, until either there exists a free edge between W and the new, extended set S , or until all vertices in W reached level ℓ . We observe that if $\mathcal{D}(t-1)$ is empty, the updates ensure that one rule of (S1)-(S3) applies. Clearly, in the cases (S2) and (S3) the desired free edge indeed exists.

Next, we show that $G(t)$ is k -colorable, under the assumption that $G(t-1)$ was not saturated and that $G(t)$ fulfills (I1). Let $e = \{x, y\}$ be the edge which Mini adds to the graph. If she uses (S2), by definition $G(t)$ is k -colorable. Otherwise, she plays between S and W . Since $|S| \leq \ell$, it is sufficient to prove that $G[W \cup C]$ is $(k - \ell)$ -colorable. We first show that $G[C]$ is $(k - \ell)$ -colorable: If $C = \emptyset$, this is trivial. Else, every vertex in C has level ℓ , thus $|S| = \ell$. When executing (S3) or (S4), Mini's edge e contains one vertex of the set W . Then clearly $G[S \cup C]$ is k -colorable as it was before, and because every vertex in S reserves one color class for its own, $G[C]$ must be $(k - \ell)$ -colorable. We extend this property to $G[C \cup W]$ as follows: We are assuming (I1), thus all non-empty subsets $A \subseteq W$ satisfy

$$2|E[A]| + |E[A, C]| \leq 2\phi(A, t) \leq 2\alpha(A) = 2 \sum_{v \in A} \ell(v) \leq 2\ell|A| < (k - \ell)|A|.$$

Then Lemma 2.2 proves that $G[C \cup W]$ is $(k - \ell)$ -colorable and indeed, Mini can insert the desired free edge.

We continue with Invariant (I1). Again, for (S2) this is trivial. In the case (S3), $\mathcal{D}(t-1)$ is empty and because Mini plays between S and W , for every set $A \subseteq W$ its potential is non-increasing, which already implies $\mathcal{D}(t) = \emptyset$. Finally for (S4), Lemma 2.3 (a) yields again that the considered set A_0 is the intersection of all sets $A \in \mathcal{D}(t-1)$. Let $A \in \mathcal{D}(t-1)$. It holds $t > 1$ and by induction hypothesis, $\mathcal{D}(t-2)$ was empty. Therefore, before Mini inserts her edge e we have $\alpha(A) + 1 = \phi(A, t-1)$. With her answer, Mini increases the level of one vertex in $v \in A_0 \subseteq A$, but the potential of the set A remains the

same, i.e., $\phi(A, t) = \phi(A, t - 1)$. Then for $G(t)$ we get $\alpha(A) = \phi(A, t)$, which proves (I1).

It remains (I2). Above we have proven that $G[C \cup W]$ is $(k - \ell)$ -colorable. Hence if $|S| < \ell$, $G(t)$ is $(k - 1)$ -colorable and clearly not saturated. If $|S| = \ell$, then by assumption there exists a free edge $e' = \{x', y'\}$ where $x' \in S$ and $y' \in W$. Let $G(t + 1) = G(t) \cup \{e'\}$. By (I1) the set $\mathcal{D}(t)$ is empty, and arguing as above for (S3) we see that $\mathcal{D}(t + 1)$ is empty as well, and Lemma 2.2 implies that $G(t + 1)$ is k -colorable and $G(t)$ was not saturated. \square

5 The Four Color Game

In this section we consider the special case $k = 4$ and analyze this particular game more precisely. First, note that our general strategies for Mini and Maxi are valid, and Theorem 1.2 and Theorem 1.3 imply

$$\frac{n^2}{4} \leq s(n, 4) \leq \frac{n^2}{3}(1 + o(1)).$$

In particular, the upper bound follows because Mini has a strategy such that G_{end} contains a vertex s of degree $n - 1$ (see Lemma 4.1), and this vertex reserves one color on its own. In order to show Theorem 1.4, we improve the lower bound by providing an alternative, more precise strategy for Maxi which allows her to elongate the game even more than with our general strategy from Section 3. The main idea is the same as before: Building up a collection of vertex-disjoint cliques ensures that no color class becomes too large. But in contrast to our previous strategy, Maxi proceeds "greedily": She draws such a clique always at once within successive moves and thereby covers the vertices which Mini used directly before. The resulting cliques can be K_2 , K_3 , and K_4 , depending on how Mini plays. As Maxi creates the cliques one after another, we can enumerate them and denote by V_i the vertex set covered with the i -th clique.

After this informal description, let us proceed with notations. Similar to our previous strategies, Maxi maintains a list C of completed vertices and a list W of vertices in the working part. She starts with $\emptyset \rightarrow C$ and $V \rightarrow W$. The set C will be always a collection of cliques; formally $C = \dot{\cup}_{i=1}^m V_i$, where every induced graph $G[V_i]$ is a clique of size $|V_i| \geq 2$. Then a move of Maxi contains always two parts: First we allow her to move a clique from W to C , afterwards she inserts her own edge. Whenever Maxi decides to change the sets W and C , we refer to such an event as an *update* of C . Note that we allow her to apply a sequence of multiple updates before drawing her own edge. Whenever C contains $i - 1$ cliques V_1, \dots, V_{i-1} and Maxi extends C by a new set V_i , we call this the i -th update, and define

$$\phi(i) := |E[W]| + |E[W, C]|$$

where we consider the freshly updated sets W and C . Our strategy will ensure that Maxi can frequently update the sets W and C and that thereby, the values $\phi(i)$ will be globally bounded. This indicates that during the game process, almost all vertices are either isolated or already contained in C . We start by describing a strategy for Maxi for producing cliques of size 2.

Lemma 5.1. *Suppose that the game reached a state where Maxi has the next move, C contains $i - 1$ cliques, $|W| \geq 3$ and $\alpha := \phi(W, C) \leq 5$. Then Maxi has a strategy such that*

- (i) *the i -th update follows after at most two rounds,*
- (ii) $\phi(i) \leq \max\{\alpha - 1, 1\}$, *and*
- (iii) $|V_i| = 2$.

Proof. Since $|V_i|$ is required to be only two, it is easy to build the desired clique. If $G[W]$ contains an edge $e = \{x, y\}$, Maxi can put $V_i = \{x, y\}$ and directly update $C \cup V_i \rightarrow C$, $W \setminus V_i \rightarrow W$. Hence, the i -th update follows immediately, before Maxi inserts her edge. Clearly, the number of edges in $G[W]$ decreases by one, which shows (ii) in this case.

If $G[W]$ is empty, let $x, y \in W$ such that $\deg(x) + \deg(y)$ is maximal. We require Maxi to play the edge $e = \{x, y\}$. Assume w.l.o.g. $\deg(x) \geq \deg(y)$, thus $\deg(y) \leq 2$ by our assumption on the value of α . After inserting e , clearly $G \setminus \{y\}$ is 4-colorable and $\deg(y) \leq 3$, thus every coloring of $G \setminus \{y\}$ can be extended to y and Maxi indeed can insert the edge e without violating the colorability constraint. Since $|W| \geq 3$ by assumption, it is easy to see that the graph is not yet saturated and Mini can answer by inserting another edge f . After Mini's answer, Maxi can put $V_i := \{x, y\}$ and update $C \cup V_i \rightarrow C$, $W \setminus V_i \rightarrow W$. It remains to prove (ii). We observe that if $\alpha \leq 2$, the edge e covers all edges of $E[C, W]$, therefore only f can contribute to $\phi(i)$, thus $\phi(i) \leq 1$. On the other hand, if $\alpha \geq 2$, the edge e covers at least two edges of $E[C, W]$ and we obtain $\phi(i) = \alpha - 1$. \square

Next, we give a similar lemma for the special case where it is Maxi's turn and the graph contains two isolated edges in W . Then, Maxi can use these two edges to construct a clique of size four.

Lemma 5.2. *Suppose that the game reached a state where Maxi has the next move, C contains $i - 1$ cliques, $|W| \geq 5$ and there exist two edges $e, e' \in E[W]$ which are isolated in G . Let $\alpha := \phi(W, C)$. Then Maxi has a strategy such that*

- (i) *the i -th update follows after at most eight rounds,*
- (ii) $\phi(i) \leq \alpha + 2$, *and*
- (iii) $|V_i| = 4$.

Proof. If the game process arrives at such a situation, Maxi takes the two edges $e = \{x, y\}$ and $e' = \{x', y'\}$ and within her next moves, she claims the four edges which connect e with e' and form a clique of size four on the vertex set $V_i := \{x, x', y, y'\}$. She needs four moves, or less if Mini helps. Since Mini can insert at most three additional edges meanwhile, she has no possibility to forbid one of the four edges and Maxi succeeds in creating this K_4 . Then, the graph is not yet saturated because $|W| \geq 5$, and we see that after an additional move of Mini (and in total after at most eight rounds), Maxi can update $C \cup V_i \rightarrow C$ and $W \setminus V_i \rightarrow W$, where $|V_i| = 4$. With e and e' , she removed two edges which contributed initially to $E[W]$. Since Mini could play at most four edges until the update, we obtain (ii) as claimed. \square

Finally we give a lemma which allows Maxi to play such that in the next update, she can move a triangle from W to C . Later on, we will see later that this is the usual answer of Maxi if Mini follows an optimal strategy.

Lemma 5.3. *Suppose that the game reached a state where Maxi has the next move, C contains $i - 1$ cliques, $|W| \geq 4$, at most one edge of $E[W]$ is isolated in G and $\alpha := \phi(W, C) \leq 3$. Then Maxi has a strategy such that*

- (i) *the i -th update follows after at most six rounds,*
- (ii) *$\phi(i) \leq 3$, and*
- (iii) *$|V_i| = 3$.*

Proof. If $G[W]$ contains a triangle $T = \{x, y, z\}$, Maxi can put $V_i := T$ and immediately updates $C \cup V_i \rightarrow C$, $W \setminus V_i \rightarrow W$. Then the i -th update follows immediately, before Maxi inserts her edge, and clearly the lemma holds in this case. If $G[W]$ contains a path $\{x, y, z\}$, clearly Maxi should play the edge $\{x, z\}$ in order to rapidly make a triangle on these three vertices. After the subsequent Mini move, she puts $V_i := \{x, y, z\}$ and updates $C \cup V_i \rightarrow C$, $W \setminus V_i \rightarrow W$. The triangle V_i then contains two edges which contributed to α , and (ii) follows immediately.

Now suppose that $E[W]$ is non-empty, but no two edges of $E[W]$ share a common vertex. Let $e = \{x, y\}$ be an edge in $E[W]$, and let $z \in W \setminus \{x, y\}$ with maximal degree. We propose Maxi to create a triangle on the three vertices $\{x, y, z\}$. Maxi needs to insert at most two edges. We observe that afterwards, there are at most three other edges between $\{x, y, z\}$ and $V \setminus \{x, y, z\}$ (two from before and one from Mini's move meanwhile), thus every coloring of $G \setminus \{x, y, z\}$ can be easily extended to $\{x, y, z\}$. Note that due to $|W| \geq 4$, this graph can not be saturated. After an additional round where Mini inserts an edge, Maxi puts $V_i := \{x, y, z\}$, moves it from W to C and updates W and C . For (ii), we observe by considering all potential subcases that due to the assumption $\alpha = \phi(W, C) \leq 3$, at most one edge which contributes to α is not contained in $E[C \cup \{x, y, z\}]$. However, Mini can insert at most two edges until the i -th update, which demonstrates $\phi(i) \leq 3$.

Finally let us consider the remaining case $E[W] = \emptyset$. Let $x, y \in W$ such that $\deg(x) + \deg(y)$ is maximal. We let Maxi play the edge $\{x, y\}$ and require her to wait for Mini's answer. In her next move, she should pick $z \in W \setminus \{x, y\}$ with maximal degree and use her next two edges to draw the triangle on the three vertices $V_i = \{x, y, z\}$. We need to show that she can play the desired edges without forbidding any proper 4-coloring of the graph. We let the two players insert all desired edges and afterwards prove that indeed, the graph is still 4-colorable.

Let us rename the vertices $\{x, y, z\}$ as $\{a, b, c\}$ such that at the time where the triangle is present it holds $\deg(a) \geq \deg(b) \geq \deg(c)$. There exists a proper 4-coloring of $G[V \setminus \{b, c\}]$ as Mini is required to keep the graph 4-colorable and all three edges of Maxi are incident to b or c . Next we observe that there are at most five edges between $V \setminus V_i$ and V_i (three edges which were counted by α , and two additional edges due to Mini's moves in the meantime), therefore b has at most two neighbors in $V \setminus V_i$. This allows us to extend the coloring to the vertex b . Finally $\deg(c) \leq 3$ and we see that at least one color is remaining for c , hence indeed the obtained graph is 4-colorable.

After Maxi succeeded in creating this triangle, the graph is not yet saturated because $|W| \geq 4$. Maxi watches for the answer of Mini before doing the update and moving V_i from W to C . It remains to bound $\phi(i)$. By assumption, $\alpha \leq 3$. Due to the strategy, at most one edge which was counted for α is not contained in $E[C \cup \{x, y\}]$. After this point in time, Mini has at most two turns until the update, and it follows $\phi(i) \leq 3$. \square

Having done this preparatory work, we can start proving Theorem 1.4. Recall that we only need to prove a matching lower bound on $s(n, 4)$.

Proof of Theorem 1.4. Maxi starts the game with $V \rightarrow W$ and $\emptyset \rightarrow C$. We claim that the three previous lemmas allow Maxi to play such that as long as $|W| \geq 5$, the graph is not saturated and she can frequently add new cliques to her collection. More precisely, we claim that if the collection contains $i - 1$ cliques and $|W| \geq 5$ at a point in time t , then the i -th update follows after at most eight rounds (counting moves of both players) and satisfies one of the following four conditions:

- (a) $\phi(i) \leq 3$ and $|V_i| = 3$,
- (b) $\phi(i) \leq 5$ and $|V_i| = 4$,
- (c) $\phi(i) \leq 4$ and $|V_{i-1}| = 4$,
- (d) $\phi(i) \leq 3$ and $|V_{i-2}| = 4$.

We prove this claim by induction over the updates. For the induction basis, we observe that no matter which player starts the game, before the first move of Maxi it holds $|W| \geq 5$ and $\phi(W, C) \leq 1$. Hence, by Lemma 5.3 she has a strategy such that after at most six rounds, she can do the first update by moving a triangle from W to C . This first update satisfies $\phi(1) \leq 3$ and thus Condition (a).

The induction step follows directly from the previous lemmas. Suppose first that the last update satisfies either (a) or (d). Then we distinguish two cases: If $E[W]$ contains two edges which are isolated in G , by Lemma 5.2 Maxi has a fast strategy to create a clique V_i of size four. Then the i -th update follows after at most eight rounds and satisfies $\phi(i) \leq 5$, i.e., (b) is fulfilled. In the other case where $E[W]$ does not contain a pair of isolated edges, by Lemma 5.3 Maxi can play such that the i -th update satisfies (a). If V_{i-1} satisfies (b), the last update was a clique of size four. Here, we let Maxi play such that the next clique is only a single edge (i.e. $|V_i| = 2$). By Lemma 5.1, Maxi is able to do so such that the i -th update satisfies (c). Finally, in the case (c) by assumption $|V_{i-2}| = 4$. Again, we require Maxi to play such that $|V_i| = 2$, and Lemma 5.1 establishes (d).

We see that as long as $|W| \geq 5$, the game does not stop and Maxi has a strategy such that at least one additional update will follow. Let \hat{t} be the first point in time where Maxi updates C and W such that $|W| < 5$ afterwards. At this point, Maxi created a collection of cliques which cover n' vertices, where $n - n' < 5$. From this point on, we let Maxi play arbitrarily until the graph is saturated. In the reminder we prove that with the given strategy, Maxi ensures that the final, saturated graph will contain at least $\frac{n^2}{3} + O(n)$ edges.

At time \tilde{t} , we have $C = \dot{\cup}_{i=1}^m V_i$. Let a_4 denote the number of cliques of size 4 in this collection, and similarly a_3 the number of triangles and a_2 the number of single edges. By definition we have $a_2 + a_3 + a_4 = m$. Moreover it holds

$$2a_2 + 3a_3 + 4a_4 = n' > n - 5.$$

Next we observe that for every clique V_i where $|V_i| = 2$, the i -th update satisfies either (c) or (d). Hence either $|V_{i-1}| = 4$ or $|V_{i-2}| = 4$, and we deduce

$$a_2 \leq 2a_4.$$

At the end of the game, there exists exactly one proper 4-coloring of G_{end} . Let C_1, \dots, C_4 be the corresponding four coloring classes. Clearly every clique of size four of the collection will contribute one vertex to every color class. Next every triangle of the collection contributes one vertex to three different color classes. In order to lower-bound the size of G_{end} , we apply Lemma 2.1 repeatedly and see that this number is the smallest if all triangles contribute to the same three color classes (say C_1, C_2, C_3). Similarly, in the worst-case all K_2 in the collection contribute to the same two color classes (say C_1 and C_2). Finally, we assume that the remaining $n - n'$ vertices not covered by the cliques are all contained in C_1 . In this worst-case, we have $|C_4| = a_4$, $|C_3| = a_3 + a_4$ and $|C_2| = a_2 + a_3 + a_4$. Using $|C_1| - |C_2| = n - n' = \mathcal{O}(1)$, we obtain that the total number of edges in G_{end} is at least

$$|C_2|^2 + 2 \cdot |C_2| \cdot (|C_3| + |C_4|) + |C_3| \cdot |C_4| + \mathcal{O}(n).$$

Minimizing this number subject to the boundary conditions on a_2 , a_3 , and a_4 is a quadratic optimization problem. For every fixed value a_3 , the number of edges will be minimized when a_2 is maximal with respect to a_4 , because this makes G_{end} as unbalanced as possible. In this case, we have $a_2 = 2a_4$, which eliminates one variable, and we deduce

$$a_2 = \frac{n' - 3a_3}{4} \quad \text{and} \quad a_4 = \frac{n' - 3a_3}{8}.$$

Using the formulas for $|C_2|$, $|C_3|$, and $|C_4|$, the calculation shows that for a fixed value a_3 , the number of edges in G_{end} is at least

$$\frac{1}{32} (11n'^2 + 2a_3n' - 9a_3^2) + \mathcal{O}(n).$$

The leading term is concave in a_3 and obtains its minimum at the boundary, therefore $a_3 \in \{0, n'/3\}$ in the worst-case. Let us shortly compare the two occurring cases. If $a_3 = 0$, then Maxi played always a combination of one K_4 and two K_2 , and the analysis yields $t_{end} = 11n^2/32 + \mathcal{O}(n)$. On the other hand, if $a_3 = n'/3$, then Maxi played only triangles, which results in $t_{end} = n^2/3 + \mathcal{O}(n)$. With the second configuration, the game stops earlier, and we deduce that $a_3 = n'/3$ minimizes the total number of edges in G_{end} . We conclude that in this case, $t_{end} \geq n^2/3 + \mathcal{O}(n)$, which lower-bounds the score of the considered saturation game. \square

Remark 5.4. *With our results we determined that $s(n, 4) = n^2/3 + \mathcal{O}(n)$. However, having the analysis on hand we can even describe the game process*

precisely when both players follow an optimal strategy: Applying Lemma 4.1, Mini always uses the same vertex s and draws a star with center s . Meanwhile, Maxi executes the strategy provided with Lemma 5.3 and covers the leafs of this star greedily with a collection of triangles. Both players are equally fast in completing their tasks, both insert $n + \mathcal{O}(1)$ edges in order to complete their tasks. At the end of this phase, $\deg(s) = n - 1$ and thus s reserves one color class on its own. On the other hand, the triangles guarantee that all other vertices are divided into three equally large color classes. This already determines the outcome of the game and the players spend the remaining time by filling the graph arbitrarily with edges until the graph is saturated.

6 Concluding remarks

In order to prove our first two results, we provided general strategies for both players, which turned out to be almost optimal. In Section 5 we have seen that at least in the case $k = 4$ it is possible to improve the general strategy for Maxi such that the game lasts even longer. We think that this is true in general and that Theorem 1.2 can be further improved with a more involved analysis and more complex strategies. However, we guess that it is challenging to determine the precise score of the colorability saturation game in general. This seems to be true even for the case $k = 3$ for which we claim that $s(n, 3) = 21n^2/64 + \mathcal{O}(n)$ (thus matching the upper-bound provided in [6]) but think that proving the lower bound requires tedious case distinctions.

Let H be a fixed graph and let $\mathcal{F}_H = \{G \mid G \text{ contains a copy of } H\}$. As we have seen in the introduction, very little is known about the saturation game (n, \mathcal{F}_H) , even for $H = K_3$. One particular example is the Hamiltonian saturation game (n, C_n) , for which it is conjectured $s(n, C_n) = \Theta(n^2)$. We hope that in near future, the knowledge on this interesting class of combinatorial games can be improved.

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